Generalized difference-form contests

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Abstract.

The present paper analyzes multi-player contests where participants compete for a valuable prize and their probability of victory depends on the difference between their effective efforts. These difference-form contests have appealing properties but remain largely understudied due to the non-existence of pure-strategy equilibria and the preemption effect they display (e.g. Che and Gale, 2000). We show that these features rest critically on the assumption of full linearity. Pure strategy equilibria with multiple active contestants exist under mild conditions as soon as full linearity is assumed away. In addition, we show that symmetric difference-form contests are equilibrium equivalent to rank-order tournaments à la Lazear and Rosen (1981) and characterize the level of total expenditures as a function of the heterogeneity in participants' valuations of victory.

Keywords: Contests, Contest success function, Inequality.
JEL codes: D31, D63, D72, D74.

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1. Introduction

Contests are present in many areas of human affairs. In military conflicts, patent races, tournaments within organizations and sport competitions, participants strive to obtain a valuable object, rent or award. In these settings, contenders often expend staggering amounts of resources. Economists are naturally interested in contests both because of their allocative implications and their impact on social welfare.[1]

The outcome of real-world contests is typically stochastic. Contest theory has modelled this randomness by assuming specific functional forms describing the rule allocating the prize contested based on the contenders' effort to acquire it. One family of these *success functions* is the ubiquitous Tullock or *ratio-form* (Tullock, 1967; 1980), which assumes that winning probabilities are proportional to contenders' efforts.[2] Another, less frequently used, family of success functions assumes that participants' winning probabilities depend on the difference between their efforts. These *difference-form* contests (Hirshleifer, 1989; 1991) present two distinctive features which can be appealing in many applications.

The first one is that contenders can enjoy a positive winning probability despite making zero effort.[3] Hirshleifer (1991; 2000) argued that this feature appears in contests with severe frictions, such as naval warfare[4] or union-management conflicts because, even when the union is very powerful, it is in its interest to keep the firm in business. By the same token, difference-form success functions are also well-suited to study lobbying within organizations (Milgrom, 1988) or federations (Wärneryd, 1998) where passive actors may still receive transfers from their managers or federal governments.

The second distinctive feature of difference-form contests is that a player can win with certainty if she overpowers her rivals by a high enough margin.[5] This is the case when one participant is overwhelmingly superior to the other in military conflicts (e.g. the US invasion of the tiny island of Grenada) or to all other candidates in hiring competitions (e.g. a Nobel prize laureate applying for a job in a average academic department). Because of this feature, difference-form contests are closely related to all-pay auctions.[6]

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[1] For excellent surveys of the contest literature see Konrad (2009) and Corchón and Serena (2018).

[2] See Skaperdas (1996) for an axiomatic study of the ratio-form.

[3] In contrast, under the standard ratio-form, such contender would lose with certainty unless every other contender bid zero.

[4] Severe storms almost obliterated the Persian navy in 480 BC, the Mongol navy in 1281 and the Spanish Armada in 1588; as a result, Greece, Japan and England respectively prevailed despite exerting virtually zero effort.

[5] Under the ratio form, victory is certain only when one contender makes *any positive effort* whilst the rest of contenders make zero effort.

[6] Che and Gale (2000) show that the mixed-strategy equilibria of two-player linear difference-form contests converge to that of the all-pay auction as the success function becomes infinitely sensitive

However, despite their appealing properties and their well-studied microfoundations[7], difference-form contests remain under-studied.[8] For instance, no paper has investigated yet the equilibria of these contests with more than two players. This is mainly due to two reasons. First, generalizing the difference-form success function to a multi-player setting is non-trivial. We solved this issue in a previous paper (Cubel and Sanchez-Pages, 2016) where we axiomatically characterized such functional form. Second, early results showed that difference-form contests display two unsavory features when contenders are heterogeneous: A pure strategy equilibrium fails to exist when one player values victory sufficiently more than the other and, when it exists, at most one contestant is active (Baik, 1998). Che and Gale (2000) conjectured that this *preemption effect* is the result of the restriction to pure-strategy equilibria.[9]

In the present paper, we generalise the difference-form contests studied in Che and Gale (2000) to more than two players and beyond full linearity. We show that, under quite mild conditions, a pure strategy equilibrium of the generalized difference-form contest exist with multiple contestants expending positive effort. In addition, we provide a thorough study of the properties of these contests. First, we show that there exists an equilibrium equivalence between symmetric difference-form contests and rank-order tournaments à la Lazear and Rosen (1981). Second, we characterize the equilibrium level of total expenditures as a function of the inequality of players' valuations of victory. Finally, we explore the mixed equilibria that emerge in the region of the parameter space where these contests have no pure strategy equilibria.

The remainder of the paper is organized as follows. In the next section we introduce the difference-form contest success function. Section 3 provides preliminary results connecting our paper with Che and Gale (2000). We provide plenty of examples throughout to facilitate understanding. Section 4 constitutes the main part of the analysis and characterizes the existence of pure strategy equilibria where multiple contenders are active. Section 5 explores the relationship between heterogeneity and total contest expenditures and studies mixed-strategies. Section 6 concludes.

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to the difference of contenders' efforts.

[7] Gersbach and Haller (2009) show that a difference-form *contest success function* (CSF) is the result of an intra-household bargaining game. Corchon and Dahm (2010, 2011), Polishchuk and Tonis (2013) and Beviá and Corchon (2019) microfound the difference-form CSF as the outcome of optimal mechanism design. Skaperdas and Vaydia (2012) show that a difference-form CSF can be derived from a Bayesian framework.

[8] Applications of difference-form sucess functions include Bush and Mayer (1974), Garfinkel (1990), Levine and Smith (1995), Grossman and Helpman (1996), Rohner (2006), Besley and Persson (2008, 2009), Gartzke and Rohner (2011), Munster and Staal (2011), Cardona and Rubi-Barcelo (2016) and Skaperdas, Toukan and Vaidya (2016).

[9] "[I]t is unclear whether this passivity is a result of the restriction to pure-strategy equilibria," (Che and Gale, 2000, p. 24).

2. Basics

Consider a community of $n \geq 2$ individuals indexed by i. These individuals are engaged in a competition only one of them can win. Examples include R&D races, military conflicts, labour tournaments, political campaigns or sport contests. Each contestant spends effort $x_i \geq 0$ in order to alter in their favor the outcome of the contest. Denote by $\mathbf{x} = \{x_1, ..., x_n\}$ the vector of individual efforts. We will say that a player is *active* if $x_i > 0$ and inactive otherwise.

2.1. The generalized difference-form success function. A *contest success function* (CSF) $p : R_+^n \rightarrow \Delta^n$ maps each vector of individual efforts $\mathbf{x}$ into a vector of individual winning probabilities.

In Che and Gale (2000), the two-player CSF was of the form

$$p_i(\mathbf{x}) = \max\{\min\{\frac{1}{2} + \frac{\beta}{2}(x_i - x_j), 1\}, 0\}. \tag{1}$$

Note that this function imposes bounds to ensure $p_1(\mathbf{x})+p_2(\mathbf{x}) = 1$ and $p_i(\mathbf{x}) \geq 0$. Otherwise, if player i's effort were low enough, i.e. $x_i < x_j - \frac{1}{\beta}$, or high enough, i.e. $x_i > x_j + \frac{1}{\beta}$, her winning probability would become negative or step above one respectively. These bounds also imply that the success function satisfies the two aforementioned features, that is, a player is not necessarily bound for defeat if she exerts zero effort and any player can guarantee her victory if she overpowers her opponent by a sufficiently large margin.

Generalizing this success function beyond the two-player case is not trivial if the success function is to maintain these features whilst generating a proper probability distribution across contenders. In a previous paper of ours (Cubel and Sanchez-Pages, 2016), we axiomatized one functional form that satisfies all these properties and generalizes the Che and Gale (2000) success function to the n-player case. We next introduce it.

Let us first order the elements of the vector of efforts in $\mathbf{x}$ such that $x_i \geq x_{i+1}$ for $i = 1, ..., n-1$. The success function is:

$$p_i(\mathbf{x}) = \begin{cases} \frac{1}{n^*} + f(x_i) - \frac{1}{n^*}\sum_{j=1}^{n^*} f(x_j) & \text{for } i = 1, ..., n^* \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $f : R_+ \rightarrow R$ is a strictly increasing *impact function* and n^* is the largest integer such that

$$\frac{1}{n^*} + f(x_{n^*}) - \frac{1}{n^*}\sum_{j=1}^{n^*} f(x_j) > 0 \;\Rightarrow\; f(x_{n^*}) > \frac{1}{n^*-1}\left(\sum_{j=1}^{n^*-1} f(x_j) - 1\right). \tag{3}$$

For the sake of clarity, it is important to examine this functional form in more detail before proceeding any further.

The first remarkable feature of the CSF in (2) is that not all players enjoy a positive winning probability for a given $\mathbf{x}$; only the players with the n^* highest impacts where n^* is determined as in (3). One player attains victory with certainty, i.e. $n^* = 1$, when $f(x_1) > f(x_2) + 1$. On the other hand, all players enjoy a positive winning probability, i.e. $n^* = n$, when

$$\frac{1}{n} + f(x_n) - \frac{1}{n}\sum_{j=1}^{n} f(x_j) > 0.$$

When there are two players and the impact function is of the form $f(x_i) = \beta x_i$, the difference-form CSF in (2) boils down to the CSF in Che and Gale (2000) where the definition of n^* in (3) replaces the bounds imposed in (1). Our functional form thus generalizes their success function to the case of n players.

The second feature of the generalized difference-form CSF is that the winning probability of a player $i \leq n^*$ is a function of the difference between her impact and the average impact of all players who enjoy a positive winning probability. If all players $i \leq n^*$ have the same impact (e.g. zero), their winning probability is just $\frac{1}{n^*}$.

The third main feature of this CSF is that, unlike the one in Che and Gale (2000), it does not impose a linear relationship between effort and impact.[10] As Cubel and Sanchez-Pages (2016) show, different impact functions can accommodate different types of effort invariance, a property which might be desirable in certain applications. More specifically, the linear impact function *à la* Che and Gale (2000) is the only one that ensures that the contest success function in (2) is *translation invariant*; that is, that winning probabilities do not change when all efforts increase by the same fixed amount.[11] This is a desirable property when the prize is allocated based on an absolute criterion, such as in promotion contests where a worker's performance must be above his colleagues'. On the other hand, the logarithmic impact function, i.e. $f(x_i) = \beta \ln x_i$, is the only function which ensures that winning probabilities are *scale invariant* (i.e. homogeneous of degree zero), a desirable property when relative efforts matter or when efforts are measured in monetary units.

An example will be useful at this point to fix ideas before proceeding any further.

Example 1: Linear impact. Suppose $n = 3$. Fix the vector of efforts to be $\mathbf{x} = (2, 1, 0)$ and the impact function to be linear, i.e. $f(x_i) = \beta x_i$. The CSF is thus translation invariant. Let us first determine n^* given $\mathbf{x}$. It is the case that $n^* = 3$ if

[10] The literature on difference-form contests has followed this assumption with the exception of Skaperdas and Vaidya (2012), Polishchuk and Tonis (2013) and Skaperdas et al. (2016).

[11] Hwang (2012) calls this property zero elasticity of augmentation.

and only if

$$\frac{1}{3} + \beta x_3 - \frac{\beta}{3}\sum_{i=1}^{3} x_i = \frac{1}{3} - \beta > 0,$$

that is, if and only if $\beta < \frac{1}{3}$, which in turn implies $\mathbf{p} = (\frac{1}{3} + \beta, \frac{1}{3}, \frac{1}{3} - \beta))$. It is the case that $n^* = 2$ if and only if $\beta \geq \frac{1}{3}$ and

$$\frac{1}{2} + \beta x_2 - \frac{\beta}{2}\sum_{i=1}^{2} x_i = \frac{1-\beta}{2} > 0,$$

that is, if and only if $\beta \in [\frac{1}{3}, 1)$. Then $\mathbf{p} = (\frac{1+\beta}{2}, \frac{1-\beta}{2}, 0)$. Finally, player 1 attains a sure victory, i.e. $n^* = 1$ and $\mathbf{p} = (1, 0, 0)$, when $\beta \geq 1$. Note that the effectiveness of effort β is crucial to determine the set of contenders with positive winning probability; as the impact function becomes more sensitive to disparities in efforts, i.e. as β increases, more players are bound to lose the contest with certainty.

2.2. The contest game. Efforts in the contest are costly. Their cost is given by a non-negative, strictly increasing, weakly convex and twice differentiable function $c : R_+ \rightarrow R_+$ which is assumed to satisfy $c(0) = 0$. Like the vast majority of the contest theory literature, Che and Gale (2000) assume the cost of effort to be linear. Whilst this is a convenient assumption that makes contest games more tractable, it is unsatisfactory for two reasons. Firstly, because the marginal cost of effort is likely to increase rapidly when effort requires time, foregone production possibilities or when capital markets are imperfect. Second, because this seemingly innocuous assumption drives results such as the *group-size paradox* (Esteban and Ray, 2001) and the *preemption effect* result in difference-form contests (Baik, 1998; Che and Gale, 2000), as we will show below.

Individuals value their victory in the contest by $v_i \geq 0$. This is the value contender i attaches to the prize, rent or award contenders are competing for. Without loss of generality, let us order players decreasingly by their valuation of victory, i.e. $v_1 \geq v_2 \geq ... \geq v_n$. Note that this ordering, together with how we laid out the success function in (2) means that we are implicitly assuming that individuals with higher valuations of victory make higher efforts. As it will become clear in the analysis below, this must be the case in equilibrium.

To conclude, individuals choose their contest effort x_i in order to maximize

$$u_i(\mathbf{x}) = p_i(\mathbf{x})v_i - c(x_i), \tag{4}$$

taking as given the effort of their opponents. Note that we are normalizing the valuation of defeat to zero; this is without loss of generality as v_i can be interpreted also as the difference between the valuations of victory and defeat.

We devote the following sections to characterize the Nash equilibria of this contest game. We will refer to this simply as the equilibrium. Before that, let us state a result which will prove useful in the ensuing analysis.

Denote by n^{**} the amount of active players under effort profile $\mathbf{x}$. Note that under a standard Tullock-type CSF it is always the case that $n^* = n^{**}$. The next Lemma describes the relationship between n^{**} and n^* under the difference-form CSF.

Lemma 1. *For any effort vector* $\mathbf{x}$ *such that* $n^{**} < n$ *it must be that* $n^* \in \{n^{**}, n\}$.

Proof. Obviously, it cannot be that an active player enjoys a zero winning probability in equilibrium, i.e. $n^* < n^{**}$. If that were the case, that player would be better off by remaining inactive. Hence, $n^* \geq n^{**}$. Assume now that $n^* \in (n^{**}, n)$ so that $p_i(\mathbf{x}) > 0$ for some contestant $i \in (n^{**}, n)$. For contender n^* it thus must be that

$$\begin{aligned} \frac{1}{n^*} + f(0) - \frac{1}{n^*}\sum_{j=1}^{n^{**}} f(x_j) - \frac{1}{n^*}\sum_{j=n^{**}+1}^{n^*} f(0) &> 0 \Leftrightarrow \\ (n^{**}+1)f(0) &> \sum_{j=1}^{n^{**}} f(x_j) - 1, \end{aligned}$$

but for contender $n^* + 1$ it must hold that

$$\begin{aligned} 0 &> \frac{1}{n^*+1} + f(0) - \frac{1}{n^*+1}\sum_{j=1}^{n^{**}} f(x_j) - \frac{1}{n^*+1}\sum_{j=n^{**}+1}^{n^*+1} f(0) \Leftrightarrow \\ (n^{**}+1)f(0) &< \sum_{j=1}^{n^{**}} f(x_j) - 1, \end{aligned}$$

leading to a contradiction. ■

3. First results

In this section, we study two basic versions of the contest game set up above. First, we explore the fully linear case which generalizes Che and Gale (2000) to n players. Second, we relax the full linearity assumption but assume instead that players value victory equally. There we show that an arbitrarily small departure from the fully linear case restores the existence of pure strategy equilibria free of the preemption effect. Before concluding the section, we show that there is an equilibrium equivalence between symmetric generalized difference-form contests and rank-order tournaments à la Lazear and Rosen (1981).

3.1. Full linearity. The set up in Che and Gale (2000) corresponds to the case where both the impact function and the cost function are linear, i.e. $f(x_i) = \beta x_i$ and $c(x_i) = x_i$. Assuming that player i is active in equilibrium, the first order condition of her maximization problem is simply

$$\frac{\partial u_i}{\partial x_i} = \beta \frac{n^* - 1}{n^*} v_i - 1 = 0. \tag{5}$$

First, note that because the success function is separable in contenders' impacts, the optimal individual effort choice does not depend directly on the effort of her opponents. Second, observe that, for a given n^*, a player is either willing to supply an arbitrarily large amount of effort or zero effort. Whether it is one case or the other depends on the player's valuation satisfying

$$v_i \geq \frac{n^*}{\beta(n^* - 1)}. \tag{6}$$

If this condition holds, the marginal benefit of effort exceeds its marginal cost and the contestant is willing to exert as much effort as necessary to win the contest with probability one. This implies that no equilibrium in pure strategies can exist when two players have a sufficiently high valuation of victory. On the other hand, if player 1 does not value victory enough, the only possible equilibrium is one where no player is active. These intuitive results are formally stated in the following proposition.

Proposition 1 [Equilibria of the linear case]. *If $v_1 < \frac{n}{\beta(n-1)}$ no contender is active in equilibrium. Otherwise, a pure strategy equilibrium exists if and only if $v_2 < \frac{2}{\beta}$. In that equilibrium, only player 1 is active and wins the contest with probability one.*

Proof. Consider first the case where $v_1 < \frac{n}{\beta(n-1)}$. There it is clear that all agents being inactive is an equilibrium. This equilibrium is unique because $\frac{n^*}{\beta(n^*-1)}$ is decreasing in n^* so if $v_1 < \frac{n}{\beta(n-1)}$, no player has any incentive to make positive effort for any $n^* \leq n$.

Consider now the case where $v_1 \geq \frac{n}{\beta(n-1)}$. Assume that in equilibrium at least one player exerts positive effort and $n^* \geq 2$. Note that if player i makes positive effort in that equilibrium then all players $j < i$ should also exert positive effort since (6) holds for them too. So if $n^* \geq 2$ it must be that player 1 is active together with all players for who $v_i \geq \frac{n^*}{\beta(n^*-1)}$. But this cannot be a pure strategy equilibrium because then player 1 could profitably deviate by increasing her effort, at least up to the point where the number of contenders with positive probability drops from n^* to $n^* - 1$. Therefore, there are only two remaining possible equilibria. Either one where all players are inactive and thus $n^* = n$, or one where player 1 is the only active player and $n^* = 1$, as recall that it cannot be an equilibrium with $n^* = 1$ where at least two players exert positive effort.

Consider the candidate equilibrium where all players remain inactive. Player 1 earns there a payoff of $\frac{v_1}{n}$. She can choose to deviate and earn payoff $u_1 = \max\{1, (\frac{1}{n} + \beta\frac{n-1}{n}x_1)\}v_1 - x_1$ instead. If $v_1 > \frac{n}{\beta(n-1)}$, player 1's best deviation is $x_1 = \frac{1}{\beta}$ and her resulting payoff is $v_1 - \frac{1}{\beta}$. That is a profitable deviation given that $v_1 - \frac{1}{\beta} > \frac{v_1}{n}$ holds when $v_1 > \frac{n}{\beta(n-1)}$. So we are left with the candidate equilibrium where only player 1 is active and wins the contest with probability one. Again, it must be that $x_1 = \frac{1}{\beta}$. Consider a potential deviation x_2' by player 2. Note that if player 2 increases her effort, even slightly, then $n^* = 2$. Hence, the derivative of player 2's payoff function at $x_2 = 0$ is

$$u_2'(\frac{1}{\beta}, 0) = v_2\frac{\beta}{2} - 1,$$

which is non-negative if and only if $v_2 \geq \frac{2}{\beta}$. Hence, such an equilibrium can survive if and only if $v_2 < \frac{2}{\beta}$. Thus, it is the only equilibrium in pure strategies when $v_1 > \frac{n}{\beta(n-1)}$.
■

This proposition generalizes Proposition 2 in Che and Gale (2000) to the case of n-players.[12] It shows that a fully linear difference-form contest cannot have an equilibrium in pure strategies unless the player with the second highest valuation of victory remains inactive because her valuation is too low. Moreover, in any pure strategy equilibrium, at most one player is active. This is what Che and Gale (2000) call the *preemption effect*, previously observed by Hirshleifer (1989) and Baik (1998). Because the marginal cost and the marginal benefit of effort are constant, contestants with a sufficiently high valuation of victory would like to spend as much effort as possible. This breaks down any pure strategy equilibria with at least two active contenders.

Proposition 1 also implies that a pure strategy equilibrium fails to exist under full linearity when players have the same valuation of victory, i.e. $v_i = v$ for all i. Next, we show that this is no longer true as soon as we move away from full linearity.

3.2. The symmetric case. Let us now depart from full linearity and characterize the equilibrium of the contest game when players' valuations of victory are identical, i.e. $v_i = v$ for all i. We show that either none or all players are active in any pure strategy equilibrium. Then we show that this symmetric difference-form contest is equilibrium equivalent to rank-order tournaments as in Lazear and Rosen (1981).

[12] Note that when $n = 2$, conditions in Proposition 1 boil down to the conditions in the text and proof of Proposition 2 in Che and Gale (2000, p. 27) after noting that their parameter $s = \frac{\beta}{2}$.

The equilibrium. Assume that the impact function $f(x_i)$ is strictly increasing, twice differentiable and weakly concave. Assume also that the cost function $c(x_i)$ satisfies all the assumptions posited in Section 2.2. When all players share the same valuation of victory $v > 0$, the first order condition of any active player is

$$\frac{\partial u_i}{\partial x_i} = \frac{n^* - 1}{n^*} v f'(x_i) - c'(x_i) = 0. \tag{7}$$

The separability of the contest success function implies that in any pure strategy equilibrium where at least a subset of players are making positive effort all must be playing the same strategy.

Define the function
$$h(x_i) = \frac{c'(x_i)}{f'(x_i)}.$$

Given our assumptions, $h(x_i)$ is non-decreasing and satisfies $h(0) \geq 0$. In particular, if *either* $c'' > 0$ *or* $f'' < 0$ (or both), that is, if we depart from the fully linear case albeit slightly, the function $h(x_i)$ is strictly increasing and invertible and a solution to (7) exists.

Denote the solution to (7) as

$$\widetilde{x}(n^*) = \max\{0, h^{-1}(\frac{n^* - 1}{n^*} v)\}.$$

This is the optimal effort choice of contestant i when n^* players enjoy a positive winning probability. Note that this optimal choice is strictly positive if and only if $v > \frac{n^*}{n^*-1} h(0)$.

The next proposition characterizes the pure strategy equilibria when players have identical valuations.

Proposition 2 [Equilibria of the symmetric case]. *Assume that $h(x_i)$ is strictly increasing. If $v < \frac{n}{n-1} h(0)$ no contender is active in equilibrium. Otherwise, all contestants must be active in any pure strategy equilibrium and the equilibrium must be symmetric. That equilibrium exists only if*

$$v \leq \frac{n}{n-1} h(f^{-1}(f(0) + \frac{1}{n-1})) \quad \textit{or} \quad v \leq \frac{n}{n-1} h(c^{-1}(\frac{v}{n})). \tag{8}$$

Proof. It is clear that all players must remain inactive in any equilibrium if $v < \frac{n}{n-1} h(0)$ as no contestant would have an incentive to exert effort then.

Assume now that $v \geq \frac{n}{n-1} h(0)$. Suppose that there exists an equilibrium where $n^* < n$ players enjoy a positive winning probability. Then, by Lemma 1, it must be that $n^* = n^{**}$. However, this implies that for the active contestants, i.e. $i \leq n^{**}$, it must be that $v \geq \frac{n^{**}}{n^{**}-1}h(0)$, whereas for the inactive contestants, i.e. $i > n^{**}$, $v < \frac{n^{**}}{n^{**}-1}h(0)$ holds. A contradiction. By the same token, this implies that it must be that $n^{**} = n^* = n$ in any pure strategy equilibrium.

Still, $v \geq \frac{n}{n-1}h(0)$ is only a necessary condition for the existence of a symmetric pure strategy equilibrium where all contenders are active. The reason is that the level of effort in the purported interior symmetric pure-strategy equilibrium $\widetilde{x}(n) = h^{-1}(v\frac{n-1}{n})$ might be so high that a contender i could be better off by remaining inactive. That is why we need to derive further conditions ensuring that $u_i(\widetilde{\mathbf{x}}(n)) \geq u_i(0, \widetilde{\mathbf{x}}_{-i}(n))$.

First, note that $p_i(0, \widetilde{\mathbf{x}}_{-i}(n)) \geq 0$ when $f(\widetilde{x}(n)) \leq f(0) + \frac{1}{n-1}$. Now observe that $u_i'(0, \widetilde{\mathbf{x}}_{-i}(n)) = v\frac{n-1}{n}f'(0) - c'(0) \geq 0$ given that $v \geq \frac{n}{n-1}h(0)$ holds by assumption. Hence, contestant i does not prefer to deviate and become inactive in that case.

The second case emerges when $f(\widetilde{x}(n)) > f(0) + \frac{1}{n-1}$. Now, $p_i(0, \widetilde{\mathbf{x}}_{-i}(n)) = p_i'(0, \widetilde{\mathbf{x}}_{-i}(n)) = 0$ and $u_i'(0, \widetilde{\mathbf{x}}_{-i}(n^*)) \leq 0$. We must then compare the payoff from deviating, i.e. $u_i(0, \widetilde{\mathbf{x}}_{-i}(n)) = 0$, with the payoff from sticking to the purported equilibrium, i.e. $u_i(\widetilde{\mathbf{x}}(n)) = \frac{v}{n} - c(\widetilde{x}(n))$. This is equivalent to checking that

$$\widetilde{x}(n) \leq c^{-1}(\frac{v}{n}) \Leftrightarrow h^{-1}(v\frac{n-1}{n}) \leq c^{-1}(\frac{v}{n}) \Leftrightarrow v \leq \frac{n}{n-1}h(c^{-1}(\frac{v}{n})), \tag{9}$$

where note that $c^{-1}(v/n)$ leaves contestants indifferent between being inactive and being active when all players, including the contestant, exert effort $c^{-1}(v/n)$.

Recall that we are in the case where $f(\widetilde{x}(n)) > f(0) + \frac{1}{n-1}$, which in turn can be rewritten as

$$v > \frac{n}{n-1}h(f^{-1}(f(0) + \frac{1}{n-1})). \tag{10}$$

So we only need (9) to hold when (10) does as well. This yields the conditions stated in the text of the proposition. Before concluding the proof, observe that in the fully linear case $h(x_i) = \frac{1}{\beta}$ the conditions for the existence of a pure strategy equilibrium with all contestants being active cannot be satisfied generically. ■

Proposition 2 shows that a pure strategy equilibrium of the symmetric case can exist where all players are active. This result is in sharp contrast with the fully linear case, where both players remain inactive in any pure strategy equilibrium under symmetric valuations . This contrast will become even starker when we analyze the asymmetric valuations case in Section 4.

For this pure strategy equilibrium to exist, the valuation of victory should not be too high or the marginal benefit of effort should not be too high relative to its marginal cost. Otherwise, contestants would invest so aggressively, i.e. $\widetilde{x}(n)$ would be so high, that they would obtain a negative payoff in that equilibrium and would

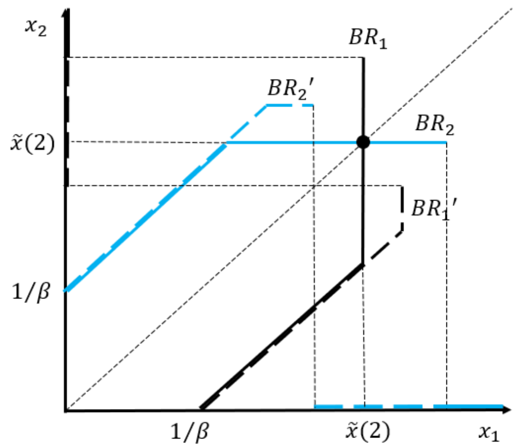

Figure 1: Best response functions for the symmetric case with $n = 2$ and linear impact.

prefer to drop out from the contest. Mixed equilibria arise then. We postpone the analysis of mixed strategies to Section 5.

Figure 1 illustrates the two scenarios characterized in Proposition 2 for the case $n = 2$ and linear impact. The lines depict the players' best response functions. The increasing sections correspond to the case where the opponent is making little effort and the best response is to make an effort $x < \widetilde{x}(2)$ that just ensures victory (hence the intercept at $\frac{1}{\beta}$). The non-increasing positive segments correspond to effort level $\widetilde{x}(2)$. When the opponent becomes very aggressive, though, the best response is to remain inactive.

The solid lines correspond to the case when the conditions in Proposition 2 are satisfied. There, players choose the optimal effort $\widetilde{x}(2)$ in equilibrium. The dashed lines correspond to the case when conditions in Proposition 2 are not satisfied. There, a pure strategy equilibrium fails to exist because $\widetilde{x}(2)$ is so high that players prefer to drop out when the opponent is exerting that effort. Note that the best response functions for the two cases overlap to a large extent.

The following example illustrates these equilibria for any n.

Example 2: Logarithmic impact function. The logarithmic impact function $f(x_i) = \beta \ln x_i$ is the only one that ensures that the success function (2) is homogeneous of degree zero (Cubel and Sanchez-Pages, 2016).

Assuming that the cost function is linear, the players' preferred effort choice becomes

$$\widetilde{x}(n) = \beta \frac{n-1}{n} v.$$

Because $h(0) = 0$, all players must be active in any pure strategy equilibrium. Recall that equilibrium existence requires that all players enjoy a non-negative payoff; otherwise they would prefer to remain inactive. Formally, this amounts simply to $\beta \leq \frac{1}{n-1}$. In other words, a full activity equilibrium exists if the sensitivity of the impact function to effort is not too high compared to the marginal cost of effort. Otherwise, competition becomes too fierce and the pure strategy equilibrium breaks down.

Equilibrium equivalence with rank-order tournaments. Next we show that difference-form success functions are very related to rank-order tournaments.[13] Following Lazear and Rosen (1981), suppose that the performance of contestant i is given by

$$y_i = x_i + \epsilon_i,$$

where ϵ_i is an idiosyncratic random additive shock. Shocks are drawn from the same distribution with density function g and differentiable cumulative function G. In the literature, it is often assumed that these shocks have zero mean. We do so below for the sake of exposition, but results in this subsection would go through if we assumed otherwise.

Contestant i wins the rank-order tournament if and only if $y_i > y_j$ for all $j \neq i$, that is, if her performance is above everybody else's. The probability of contestant i winning the tournament is

$$\begin{aligned} \Pr(i \text{ wins}) &= \Pr(\epsilon_j < x_i - x_j + \epsilon_i \ \forall j \neq i); \\ &= \int \prod_{j \neq i} G(x_i - x_j + t) dG(t). \end{aligned}$$

Assume all contestants in the tournament value the prize at v and that the cost of effort is given by the strictly increasing and differentiable function $c(x_i)$. Consider a symmetric pure strategy equilibrium $\mathbf{x}^* > \mathbf{0}$. The expected payoff of contender i from a deviation from this equilibrium is

$$u_i(x_i, \mathbf{x}^*_{-i}) = v \int G(x_i - x^* + t)^{n-1} dG(t) - c(x_i).$$

So for $\mathbf{x}^*$ to be an equilibrium it must be that the following first order condition holds:

$$v(n-1) \int G(t)^{n-2} g(t) dG(t) - c'(x^*) = 0.$$

Given our assumptions on $c(x_i)$, a solution to this equation exists and is unique. Thus, a unique symmetric equilibrium exists. We next characterize the conditions

[13] This connection was first acknowledged in Che and Gale (2000, p. 24).

under which that symmetric equilibrium is equivalent to the equilibrium of the symmetric difference-form contest characterized in Proposition 2.

Proposition 3 [Equivalence with rank-order tournaments]. *Assume that the impact function is linear and the conditions in Proposition 2 hold. Then, the pure strategy equilibrium of the symmetric difference form-contest is equivalent to the equilibrium of a rank-order tournament with an additive random shock distributed with cdf $G(t) = e^{\beta t-1}$ and support $(-\infty, \frac{1}{\beta}]$.*

Proof. First, note that when the random shock has cdf $G(t) = e^{\beta t-1}$ and support $(-\infty, \frac{1}{\beta}]$ one can write $\left.\frac{\partial u_i(x_i, \mathbf{x}^*_{-i})}{\partial x_i}\right|_{x_i=x^*} = 0$ as

$$v(n-1)\beta^2 \int_{-\infty}^{\frac{1}{\beta}} e^{n(\beta t-1)} dt = c'(x^*).$$

Solving the integral yields

$$v\beta\frac{n-1}{n} = c'(x^*),$$

which is the first order condition for the interior symmetric equilibrium in the difference-form contest with linear impact assuming that conditions in Proposition 2 hold. If $v\beta\frac{n-1}{n} < c'(0)$, all players remain inactive in the symmetric equilibrium of both games. ■

This proposition shows that if shocks follow a generalized type-III extreme value distribution[14] -also called *negative Weibull*-, with zero mean and variance $1/\beta^2$, the rank-order tournament and the symmetric difference-form contest are equilibrium equivalent. This equivalence arises naturally because the generalized type-III extreme value distribution is the limit distribution of the maximum of a sequence of independent and identically distributed random variables. In the symmetric equilibrium, the probability of winning the tournament is entirely determined by the probability of the shock being the largest among all players'. That is also why Ryvkin and Drugov (2017) find that the symmetric Tullock (ratio) contest is equilibrium equivalent to a rank-order tournament with additive shocks distributed following a type-I extreme value distribution, another maximum limit distribution.

Before closing this section, let us note that the variance of the negative Weibull is $1/\beta^2$. That is, as the sensitivity parameter β increases, noise disappears and the measure of performance becomes more accurate. In the limit, when $\beta \to \infty$, both the rank-order tournament and the difference-form contest approach the all-pay auction as the player with the highest effort wins with certainty.

[14] See Jenkinson (1955) for an early reference.

4. The asymmetric case

Let us now turn our attention to the case where contestants hold different valuations. The CSF in (2) already hints that players with lower valuations will remain inactive. We next characterize the equilibria in pure strategies where multiple heterogeneous contestants are active.

When valuations are heterogenous, the FOC associated to the optimization problem of a player who is active in equilibrium is

$$\frac{\partial u_i}{\partial x_i} = \frac{n^* - 1}{n^*} v_i f'(x_i) - c'(x_i) = 0.$$

Again, note that the separability of the contest success function implies that the solution to this problem does not depend directly on the effort of other players. This solution is

$$\widetilde{x}_i(n^*) = \max\{0, h^{-1}(\frac{n^* - 1}{n^*} v_i)\}. \tag{11}$$

We refer to the level of effort $\widetilde{x}_i(n^*)$ as the *preferred effort choice* of contestant i. This preferred effort choice is strictly positive if and only if $v_i > \frac{n^*}{n^*-1} h(0)$. It is non-decreasing in the valuation of victory v_i, implying $\widetilde{x}_i(n^*) \geq \widetilde{x}_{i+1}(n^*)$, and in n^*, i.e. $\widetilde{x}_i(n^*) \leq \widetilde{x}_i(n^* + 1)$.

Although the preferred effort choice of a player does not depend on the effort of others, her best response does. Even if $\widetilde{x}_i(n^*) > 0$, contestant i will prefer to remain inactive if the rest of contenders exert sufficiently high effort. Note also that a contender with a low valuation of victory so that $\widetilde{x}_i(n^*) = 0$ can still enjoy a positive winning probability in equilibrium if the rest of contenders have relatively low valuations and make low enough efforts.

Figure 2 illustrates these scenarios for $n = 2$ and linear impact. The solid lines are the best response functions of the two players corresponding to the case where a pure strategy equilibrium exists; players make effort $\widetilde{x}_i(2)$ in that equilibrium. The dashed best response function corresponds to the case where player 2's valuation has decreased and effort $\widetilde{x}_2(2)$ is not enough to obtain a positive payoff when player 1 chooses $\widetilde{x}_1(2)$ (note that her best response functions partially overlap). In that case, any equilibrium must be in mixed strategies. We will come back to this in Section 5.2.

Hopefully, it should have become clear to the reader at this point that the characterization of pure strategy equilibria hinges on the characterization of the number of contenders with positive winning probabilities n^* and the number of active contenders n^{**}. Given the preferred effort choices of the n^{**} active contenders when

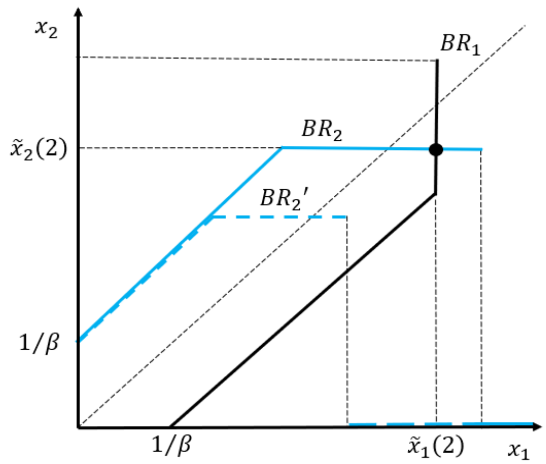

Figure 2: Figure 2: Best responses for the asymmetric case with $n = 2$ and linear impact.

n^* contenders have $p_i > 0$, it must be that precisely n^* contenders enjoy a positive winning probability under the strategy profile in which n^{**} contestants are active and exert their preferred effort choice.

Let us now characterize conditions under which such pure strategy equilibria exist.

Proposition 4 [**Pure strategy equilibria with heterogeneous players**]. *Assume $h(x_i)$ is strictly increasing and that $v_1 \geq \frac{n}{n-1}h(0)$. Then, the generalized difference-form contest admits the following pure strategy equilibria:*

(i) An equilibrium with $n^{**} = n^* = 1$ when $v_2 < 2h(0)$ and

$$f(\widetilde{x}_1(n)) \geq 1 + f(0);$$

(ii) Equilibria with $n^{**} < n^* = n$ when $v_{n^{**}+1} < \frac{n}{n-1}h(0) \leq v_{n^{**}}$ and

$$\sum_{i=1}^{n^{**}} f(\widetilde{x}_i(n)) \leq 1 + n^{**}f(0); \tag{12}$$

(iii) Equilibria with $2 \leq n^{**} = n^* \leq n$ when $v_{n^{**}+1} < \frac{n^{**}}{n^{**}-1}h(0) \leq v_{n^{**}}$ and

$$\sum_{i=1}^{n^{**}-1} f(\widetilde{x}_i(n^{**})) \leq 1+\max\{(n^{**}-1)f(0), (n^{**}-1)f(\widetilde{x}_{n^{**}}(n^{**}))-\frac{n^{**}}{v_{n^{**}}}c(\widetilde{x}_{n^{**}}(n^{**}))\}. \tag{13}$$

Proof. Player 1 has the highest incentive to deviate from an equilibrium where all players remain inactive. Her marginal benefit of effort at that equilibrium is $v_1 \frac{n-1}{n} f'(0)$, so she will deviate as long as $v_1 \geq \frac{n}{n-1} \frac{c'(0)}{f'(0)} = \frac{n}{n-1} h(0)$. This ensures that at least player 1 has an incentive to exert positive effort.

Let us now study equilibria where $n^{**} \geq 1$. We first characterize the conditions under which an equilibrium with only one active player exists, i.e. $n^* = n^{**} = 1$. Player 1's equilibrium choice should maximize then

$$u_1 = \max\{1, \frac{1}{n} + \frac{n-1}{n} f(x_1) - \frac{n-1}{n} f(0)\} v_1 - c(x_1).$$

The solution is $x_1^* = \min\{f^{-1}(1 + f(0)), \widetilde{x}_1(n)\}$. Note that $p_1^* = 1$ only when $\widetilde{x}_1(n) \geq f^{-1}(1+f(0))$, which is the same as $f(\widetilde{x}_1(n)) \geq 1 + f(0)$. In addition, we need that $v_2 < 2h(0)$; otherwise player 2 would like to become active. These conditions are summarized in part (i) of the text of the Proposition.

Let us now construct an equilibrium where $n^{**} < n^* = n$. This is an equilibrium where $n - n^{**}$ players remain inactive but all enjoy a positive winning probability. Formally, it must be that $\widetilde{x}_i(n) > 0$ for $i \leq n^{**}$ and

$$p_i = \frac{1}{n}\left(1 + n^{**} f(0) - \sum_{i=1}^{n^{**}} f(\widetilde{x}_i(n))\right) \geq 0, \tag{14}$$

for $i \geq n^{**} + 1$. In addition, player $n^{**} + 1$ should not have an incentive to become active. That is,

$$v_{n^{**}+1} < \frac{n}{n-1} h(0) \tag{15}$$

Rewriting (14), and combining it with (15) and $\widetilde{x}_{n^{**}}(n^{**}) > 0$ produces the conditions stated in part (ii) of the text of the proposition.

The last type of equilibrium is one where $2 \leq n^* = n^{**}$. For that equilibrium to exist a necessary condition is $v_{n^{**}+1} < \frac{n^{**}}{n^{**}-1} h(0)$. Otherwise, at least player $n^{**} + 1$ would deviate and become active. The second necessary condition is that $\widetilde{x}_{n^{**}}(n^{**}) > 0$, which is equivalent to $v_{n^{**}} \geq \frac{n^{**}}{n^{**}-1} h(0)$. The last condition is that player n^{**} does not prefer to deviate from $\widetilde{x}_{n^{**}}(n^{**})$ and become inactive. As in the proof of Proposition 2 we need to entertain two possibilities.

If condition

$$\sum_{i=1}^{n^{**}-1} f(\widetilde{x}_i(n^{**})) \leq 1 + (n^{**} - 1) f(0), \tag{16}$$

is satisfied, then player n^{**} enjoys a positive winning probability by being inactive when players $i < n^{**}$ make effort $\widetilde{x}_i(n^{**})$. But then $u'_{n^{**}}(0, \{\widetilde{x}_i(n^{**})\}_{i=1}^{n^{**}-1}, \{0\}_{i \geq n^{**}}) > 0$ because we have already imposed that $v_{n^{**}} \geq \frac{n^{**}}{n^{**}-1} h(0)$. This implies that $\widetilde{x}_{n^{**}}(n^{**})$ is indeed a global maximum and it is the best response to $\{\{\widetilde{x}_i(n^{**})\}_{i=1}^{n^{**}-1}, \{0\}_{i \geq n^{**}}\}$.

If condition (16) does not hold, $\widetilde{x}_{n^{**}}(n^{**})$ might be just a local maximum. The reason is that the rest of active players are so aggressive that player n^{**} may obtain a negative payoff by making effort $\widetilde{x}_{n^{**}}(n^{**})$. Her preferred effort choice is a best response if and only if

$$u_{n^{**}}(\{\widetilde{x}_i(n^{**})\}_{i=1}^{n^{**}}, \{0\}_{i \geq n^{**}}) \geq 0 \Leftrightarrow$$

$$\sum_{i=1}^{n^{**}-1} f(\widetilde{x}_i(n^{**})) \leq 1 + (n^{**}-1) f(\widetilde{x}_{n^{**}}(n^{**})) - \frac{n^{**}}{v_{n^{**}}} c(\widetilde{x}_{n^{**}}(n^{**})).$$

All these conditions jointly characterize the equilibrium in part (*iii*) of the text of the Proposition. ■

Proposition 4 shows that pure strategy equilibria with multiple players can be of two types: First, an equilibrium where all contestants enjoy a positive winning probability but only a subset of them are active. This equilibrium exists when the highest valuations are low enough so that the rest of contestants still enjoy a positive winning probability when these high-valuation players make their preferred effort choice. Second, an equilibrium where the set of active players coincides with the set of contestants who have a positive winning probability, i.e. $n^{**} = n^*$. That equilibrium exists when the players with the highest valuations make enough effort to preempt the rest from becoming active. In addition, the set of active players must have relatively similar valuations among them so that none of them prefer to drop out. These equilibria are realistic in the sense that contenders with lower valuations remain inactive whereas those with higher valuations participate in the contest.

In addition, Proposition 4 reiterates that the full preemption result in Che and Gale (2000) breaks down as soon as full linearity is dropped. Some, even all, contenders can be active in equilibrium if they have high enough valuations relative to those of their opponents. To see this, consider the following corollary to Proposition 4.

Corollary 1: *If* $h(0) = 0$, *then* $n^{**} = n$ *in any pure strategy equilibrium.*

For full activity to be an equilibrium when $h(0) = 0$, v_n must be high enough so that effort $\widetilde{x}_n(n)$ awards player n a positive payoff when competing against the rest of players. Note the contrast with the pure strategy equilibria of the fully linear case characterized in Proposition 1. There player 2 had to value victory not too much, i.e. $v_2 < \frac{2}{\beta}$, for a pure strategy equilibrium (with preemption) to exist whereas here a pure strategy equilibrium (with full activity) exists only when v_n is high enough.

Let us illustrate this contrast and the pure strategy equilibria characterized in Proposition 4 with the following example.

Example 3: Exponential cost function. Consider the cost function

$$c(x_i) = \frac{e^{\phi x_i} - 1}{\phi},$$

where $\phi \geq 0$. This function encompasses as a particular case the linear cost function when $\phi = 0$. Moreover, $c'(0) = 1$ for all ϕ.

When, in addition, the impact function is linear, the preferred effort of player i is

$$\widetilde{x}_i(n^*) = \max\{0, \frac{1}{\phi} \ln \beta \frac{n^* - 1}{n^*} v_i\}.$$

An equilibrium with $n^{**} = 3$ active players exists if $v_3 \geq \frac{3}{2\beta}$ and

$$G_2 \leq v_3 e^{\frac{\phi}{2\beta} + \frac{3}{2\beta v_3} - 1},$$

where $G_2 = \sqrt{v_1 v_2}$ is the geometric mean of the valuations of the contestants with the two highest valuations. In other words, a pure strategy equilibrium with full activity exists as long as the two contestant with the highest valuations value victory not too highly relative to the third player.[15] The geometric mean will reappear in our analysis of inequality and total contest expenditures in Section 5.1.

An equilibrium with $n^{**} = 2$ and $n^* = 3$ exists when $v_3 < \frac{3}{2\beta} \leq v_2$ and

$$G_2 < \frac{3}{2\beta} e^{\frac{\phi}{2\beta}}.$$

In words, this equilibrium exists if and only if the two players with the highest valuations do not value victory so much as to ensure the defeat of the third one when she remains inactive.

Figure 3 illustrates the regions of the parameter space for the case $n = 3$ provided that $v_3 < \frac{3}{2\beta}$. In the white areas, the pure strategy equilibrium breaks down. In the white area on the left, the equilibrium must be in mixed strategies because player 1 and 2 value victory enough to defeat player 3 for sure, but player 2 has a relatively low valuation and no incentive to make a positive effort against player 1 alone. The other white area corresponds to the case where player 2 has a high enough valuation so her preferred effort choice is positive when competing against player 1 but this level of effort is not high enough to secure a positive payoff. We explore the resulting mixed-strategy equilibria in the next section.

[15] Note that the right hand side of the expression is increasing in v_3 whenever $v_3 \geq \frac{3}{2\beta}$.

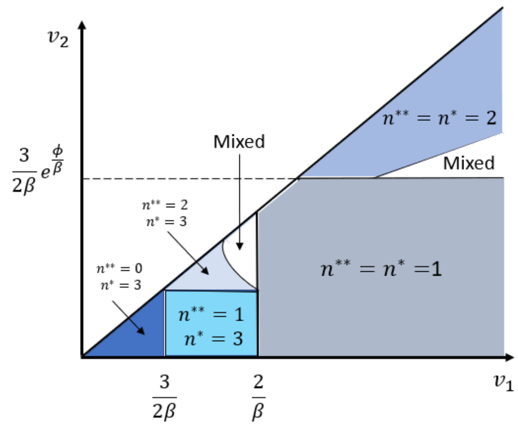

Figure 3: Equilibria for Example 3 when $v_3 < \frac{3}{2\beta}$.

5. Further results

In this section, we derive some additional results. First, we study how the level of total expenditures in the contest under the pure strategy equilibrium characterized above varies with the heterogeneity in players' valuations. Then, we explore a mixed strategy equilibrium that exists in the region of the parameter space where pure strategy equilibria fail to exist.

5.1. Heterogeneity and total contest expenditures. Next we explore how heterogeneity in players' valuations affects aggregate expenditures in difference-form contests. In Tullock contests, Cornes and Hartley (2005) showed that total expenditures decrease with heterogeneity when the cost of effort is a power function whereas Ryvkin (2013) uncovered a much more complex relationship for general cost functions depending on their curvature. Similarly, we show in what follows that the relationship between aggregate effort and heterogeneity in difference-form contests depends on the convexity or concavity of the function $h(x_i)$.

First, we need to establish a criterion to compare distributions of valuations. To that end, we employ the well-known Pigou-Dalton principle of transfers. That is, we explore changes in the valuation of one player accompanied by a change of identical magnitude but opposed sign in the valuation of another player whilst keeping their ranking in the valuation distribution unchanged. We perform a *progressive (regressive) change* when the player whose valuation decreases has a higher (lower) valuation than the contestant whose valuation increases. According to the Pigou-Dalton principle of transfers, the resulting distribution is more equal (less heterogeneous) than the original one. After performing a progressive change, the resulting distribution of valuations Lorenz dominates the original one.

The following proposition compares the total level of contest expenditures after a progressive change in valuations which leaves n^*constant.

Proposition 5 [**Heterogeneity and total effort**]. *Consider a valuation distribution* $\mathbf{v}$ *under which* n^* *players enjoy a positive winning probability and* $n^{**} \geq 2$ *out of them are active in equilibrium. Perform a progressive change in* $\mathbf{v}$ *between any two players* $i, j \leq n^{**}$ *such that* n^* *and* n^{**} *remain unchanged. The total equilibrium level of effort increases (decreases) as a result if and only if* $h(x_i)$ *is convex (concave).*

Proof. The total level of expenditures in an equilibrium with $n^{**} \geq 2$ is given by

$$\sum_{i=1}^{n^{**}} \widetilde{x}_i = \sum_{i=1}^{n^{**}} h^{-1}(\frac{n^* - 1}{n^*} v_i).$$

After a progressive change leaving n^* and n^{**} unchanged, the vector $\{h^{-1}(\frac{n^*-1}{n^*}v_i)\}_{i=1}^{n^*}$ majorizes the vector $\{h^{-1}(\frac{n^*-1}{n^*}v_i')\}_{i=1}^{n^*}$, where $\mathbf{v}'$ is the valuation distribution resulting from the progressive change. If h is convex (concave) then its inverse h^{-1} is concave (convex) and therefore

$$\sum_{i=1}^{n^*} h^{-1}(\frac{n^* - 1}{n^*} v_i) < (>) \sum_{i=1}^{n^*} h^{-1}(\frac{n^* - 1}{n^*} v_i').$$

■

The effect of small changes that make the valuation distribution more egalitarian depend on the shape of the impact and cost function. The intuition is that if the cost function is very convex or the impact function very concave, players with higher valuations exert relatively low effort in equilibrium. In that case, a progressive change makes the lower valuation player increase her effort more than the higher valuation player decreases her own.

We can use this result to study how the equilibrium number of players with positive winning probabilities changes in response to a progressive change in the distribution of valuations.

Proposition 6 [**Heterogeneity and winning probabilities**]. *Assume that the impact function is linear. Consider a valuation distribution* $\mathbf{v}$ *under which* n^* *is the equilibrium number of players with positive probability,* n^{**} *out of them are active. Perform a progressive change in* $\mathbf{v}$ *between any two players* $i, j < n^{**}$ *which leaves* n^{**} *unchanged. Denote by* n' *the number of players with positive probability under the new distribution* $\mathbf{v}'$*. Then* $n' \leq n^*$ $(n' \geq n^*)$ *when* $h(x_i)$ *is convex (concave).*

Proof. Under the valuation distribution $\mathbf{v}$ it holds that n^* is such that

$$\frac{1}{n^*} + \beta\widetilde{x}_{n^*}(n^*) - \frac{\beta}{n^*}\sum_{j=1}^{n^*}\widetilde{x}_j(n^*) \quad > \quad 0 \qquad \text{and}$$

$$\frac{1}{n^*+1} + \beta\widetilde{x}_{n^*+1}(n^*+1) - \frac{\beta}{n^*+1}\sum_{j=1}^{n^*+1}\widetilde{x}_j(n^*+1) \quad < \quad 0.$$

Note that the negative term in the first expression contains the total level of contest expenditures. Hence, we can apply Proposition 5 to establish that the left hand side of the first equation decreases when moving from $\mathbf{v}$ to $\mathbf{v}'$ when h is convex. This implies that $n' \leq n^*$. By the same token, the left hand side of the second expression increases when moving from $\mathbf{v}$ to $\mathbf{v}'$ when h is concave, so $n' \geq n^*$ as a result. ∎

The intuition of this result is similar to the one for Proposition 5: When the cost function is quite convex or the impact function very concave, a progressive change in valuations increases the level of overall competition. This makes lower valuation players less likely to secure a positive payoff with their preferred effort choice.

The following example illustrates how the shape of the h function determines the impact on the total level of contest expenditures of a more egalitarian distribution of valuations.

Example 4: Isoelastic cost. Consider the isoelastic cost function $c(x_i) = \frac{1}{1+\phi}x_i^{1+\phi}$ where $\phi \geq 0$. The case $\phi = 0$ corresponds to the linear cost case. Assume the impact function is linear. Then, $h(x_i)$ is strictly convex (concave) if $\phi > (<)1$.

Suppose there are three players with valuations $\mathbf{v} = (4, 2, \frac{2}{3})$ and $\beta = \frac{1}{2}$. Consider first the case $\phi = \frac{1}{2}$ so that $h(x_i)$ is strictly concave. In equilibrium, $\mathbf{p}^* = (\frac{11}{16}, \frac{5}{16}, 0)$. Consider now a rank-preserving progressive change in $\mathbf{v}$ such that $v_1' = 4 - \Delta$ and $v_2' = 2 + \Delta$ where $\Delta \in [0, 1]$. If $\Delta \leq \frac{1}{3}$, it is still the case that $n^* = 2$ and the total level of contest expenditures is given by $\frac{1}{16}[(v_1')^2 + (v_2')^2]$, which is decreasing in Δ. That is, total effort decreases as the valuation distribution becomes more egalitarian. If $\Delta > \frac{1}{3}$, players 1 and 2 become less competitive and the number of contenders with positive winning probability increases to $n^* = 3$.

Now assume the same $\mathbf{v}$, $\beta = \frac{3}{4}$ and $\phi = 1.9$ so $h(x_i)$ is strictly convex. In equilibrium, $\mathbf{p}^* = (0.66, 0.33, 0.01)$ and $n^* = 3$. Perform a progressive change as before. As long as $\Delta \lesssim 0.23$, $n^* = 3$ in equilibrium and the total level of contest expenditures is given by $(\frac{1}{2})^{\frac{1}{1.9}}[(v_1')^{\frac{1}{1.9}} + (v_2')^{\frac{1}{1.9}} + (v_3')^{\frac{1}{1.9}}]$, which is increasing in Δ; total effort increases as the distribution of valuations becomes more egalitarian. If $\Delta > 0.23$, players 1 and 2 become so competitive that the number of contenders with positive probability drops to $n^* = 2$.

5.2. Mixed strategies. We now turn our attention to the region of the parameter space where pure strategy equilibria fail to exist. Recall that $n^{**} = n^* \leq n$ cannot be an equilibrium when two conditions are met: First, that the valuation of player n^{**} is not high enough relative to those of players with higher valuations than her; in that case, her preferred effort choice $\widetilde{x}_{n^{**}}(n^{**})$ does not guarantee her a positive payoff and she prefers to drop out. Second, it is the case that contestant n^{**} has an incentive to become active when $n^{**}-1$ players are active, i.e. $\widetilde{x}_{n^{**}}(n^{**}-1) > 0$, so $n^{**}-1$ cannot be a pure strategy equilibrium either. Hence, the equilibrium can only be in mixed strategies.

The next lemma provides two general results about any mixed strategy equilibria of our contest game.

Lemma 2: *Assume the impact function is linear and $h(0) = 0$. In any mixed strategy equilibrium, at least one player is fully active and at most $n-2$ players remain fully inactive. Moreover, any player who puts probability mass at zero must earn a zero payoff.*

Proof. For the first part of the statement, suppose by contradiction that all players put a probability mass at zero, so the event where all players remain inactive occurs with positive probability. Then, any player could increase her effort from zero to ε and make her winning probability go from $\frac{1}{n}$ to $\frac{1}{n} + \beta\frac{n-1}{n}\varepsilon$ in that event. Since $h(0) = 0$ implies $c'(0) = 0$, this deviation is profitable. Now suppose that all but one player remain fully inactive. If that active player wins with probability $p_i < 1$, the marginal benefit of effort for all the other players $j \geq i$ is $\beta\frac{n-1}{n}v_j$ at zero, which is thus greater than $c'(0)$, so they would deviate. If the only active player is winning with probability one, it must be that $x_i = \frac{1}{\beta}$. The marginal benefit of effort for any inactive player who becomes active would then be $\frac{\beta}{2}v_j > 0$. Hence, any player would prefer to deviate and become active.

For the second statement, suppose on the contrary that the payoff of the player who puts probability mass at zero is positive. This can only occur if the expected bid of the opponents $E[\mathbf{x}^*_{-i}]$ is such that $p_i(0, E[\mathbf{x}^*_{-i}]) > 0$. This implies that the expected marginal benefit of increasing effort from zero to $\varepsilon > 0$ is positive, i.e. $\beta\frac{n^*-1}{n^*}v_i$. Because $c'(0) = 0$, such deviation would be profitable. Hence, i must be earning a zero payoff in equilibrium. ■

Che and Gale (2000) characterized two types of mixed strategy equilibria in two-player fully linear difference-form contests: An *overlapping* equilibrium where the two players put some probability mass at zero, and a *staggered* equilibrium in which the players put mass at a number of points separated by a fixed length and the player with the lowest valuation puts some mass at zero. The lemma above shows that an overlapping equilibrium cannot exist in generalized difference-form contests when $h(0) = 0$. That type of equilibrium is thus an artifact of cost linearity. When $h(0) = 0$, players have an incentive to be fully active when the rest of players put some probability mass at zero.

It is possible to write down a system of equations that characterizes a candidate staggered mixed equilibrium of the generalized difference-form contest. But unlike in Che and Gale (2000), these equations are non-linear and it is thus not possible to obtain a general existence result. Because a full characterization of mixed equilibria is out of the scope of this paper, we focus our attention to the case where the impact function is linear and the cost function is quadratic. This case illustrates well how that staggered mixed equilibria would look like: The player with the highest valuation is active with probability one, the player with the second highest valuation randomizes between her preferred effort choice and inactivity, and the rest of players remain fully inactive.

Proposition 7. *Assume impact is linear, the cost is quadratic and $c(\widetilde{x}_2(2)) < v_2$ but (13) is not satisfied for $n^{**} = 2$. Then a mixed strategy equilibrium exists which satisfies the following properties:*

(i) *Player 1 makes effort*

$$\varkappa_1 = \frac{1}{\beta} + \frac{\widetilde{x}_2(2)}{2},$$

with probability 1;

(ii) *Player 2 makes effort $\widetilde{x}_2(2)$ with probability*

$$\pi = \frac{2}{\beta v_1}\varkappa_1,$$

and remains inactive otherwise;

(iii) *Players $i = 3, ..., n$ remain fully inactive;*

(iv) *Player 1's expected payoff is $v_1(1 - \pi\frac{c(\widetilde{x}_2(2))}{v_2}) - c(\varkappa_1)$ whereas the rest of contestants earn zero.*

Proof. For the time being suppose that the contest has only two players, 1 and 2. Recall from Proposition 4 that a mixed strategy equilibrium must exist when $v_2 \geq 2h(0)$ but (13) is not satisfied for $n^{**} = 2$; for the case with linear impact and isoelastic cost, this boils down to

$$\widetilde{x}_1(2) > \frac{1}{\beta} + \max\{0, \widetilde{x}_2(2) - \frac{2c(\widetilde{x}_2(2))}{\beta v_2}\}; \tag{17}$$

$$\Leftrightarrow \quad v_1 > \frac{2}{\beta^2} + 2\frac{\widetilde{x}_2(2)}{\beta}. \tag{18}$$

In words, v_1 is so high relative to v_2 that $\widetilde{x}_2(2)$ does not guarantee player 2 a higher payoff than inactivity against $\widetilde{x}_1(2)$. But inactivity is not a best response either for player 2 either because $v_2 \geq 2h(0) = 0$ and she would like to become active when only player 1 is active.

Now, consider the strategy profile described in the text of the proposition. Player 2 makes effort $\widetilde{x}_2(2)$ with some probability π and remains inactive with the remaining probability $1-\pi$. With some abuse of notation, denote this strategy profile simply as $\varkappa_2$. On the other hand, consider a pure strategy $\varkappa_1$ for player 1 where $\varkappa_1$ is such that $p_2(\varkappa_1, \widetilde{x}_2(2)) \in [0,1]$ and $p_2(\varkappa_1, 0) = 0$. Note that we are not including the vector of zero efforts corresponding to players $i \geq 3$ as we are for the time being assuming there are only two contenders.

For the proposed profile $\{\varkappa_1, \varkappa_2\}$ to constitute a mixed strategy equilibrium, player 2 must be indifferent between exerting effort $\widetilde{x}_2(2)$ and remaining inactive. That is, $\varkappa_1$ must be such that

$$\begin{aligned} u_2(\varkappa_1, \widetilde{x}_2(2)) &= u_2(\varkappa_1, 0); \\ &\Leftrightarrow \; v_2 p_2(\varkappa_1, \widetilde{x}_2(2)) - c(\widetilde{x}_2(2)) = 0; \\ &\Leftrightarrow \; \varkappa_1 = \frac{1}{\beta} + \widetilde{x}_2(2) - \frac{\widetilde{x}_2(2)^2}{\beta v_2} = \frac{1}{\beta} + \frac{\widetilde{x}_2(2)}{2}, \end{aligned} \tag{19}$$

where the last equality comes from noting that $\widetilde{x}_2(2) = \frac{\beta}{2} v_2$.

Next we need to check that it is indeed the case that $1 \geq p_2(\varkappa_1, \widetilde{x}_2(2)) \geq 0 = p_2(\varkappa_1, 0)$ and that $\varkappa_1$ is a best response to $\varkappa_2$. The former is equivalent to

$$\max\{\frac{1}{\beta}, \widetilde{x}_2(2) - \frac{1}{\beta}\} \leq \varkappa_1 \leq \frac{1}{\beta} + \widetilde{x}_2(2).$$

Given the definition of $\varkappa_1$ in (19), the second inequality holds automatically. Note that $\varkappa_1 \geq \frac{1}{\beta}$ also holds automatically because of (19). Finally note $\varkappa_1 \geq \frac{\beta}{2} v_2 - \frac{1}{\beta}$ if and only if $c(\widetilde{x}_2(2)) \leq v_2$, which we impose.

Let us now check whether $\varkappa_1$ is indeed a best response to $\varkappa_2$. To do that, we need to show that $x_1 = \varkappa_1$ maximizes player 1's payoff function globally when player 2 follows strategy $\varkappa_2$. Before writing this payoff function, recall that when impact is linear, a player wins (loses) with certainty when her effort is at least $\frac{1}{\beta}$ units higher (lower) than her opponent's.

$$u_1(x_1, \varkappa_2) = \begin{cases} (1-\pi) v_1 p_1(x_1, 0) - c(x_1) & \text{If } x_1 < \min\{\frac{1}{\beta}, \widetilde{x}_2(2) - \frac{1}{\beta}\}; \\ (1-\pi) v_1 p_1(x_1, 0) + \pi p_1(x_1, \widetilde{x}_2(2)) v_1 - c(x_1) & \text{if } x_1 \in [\widetilde{x}_2(2) - \frac{1}{\beta}, \frac{1}{\beta}); \\ (1-\pi) v_1 - c(x_1) & \text{if } x_1 \in [\frac{1}{\beta}, \widetilde{x}_2(2) - \frac{1}{\beta}); \\ (1-\pi) v_1 + \pi p_1(x_1, \widetilde{x}_2(2)) v_1 - c(x_1) & \text{if } x_1 \in [\max\{\frac{1}{\beta}, \widetilde{x}_2(2) - \frac{1}{\beta}\}, \frac{1}{\beta} + \widetilde{x}_2(2)); \\ v_1 - c(x_1) & \text{if } x_1 \geq \frac{1}{\beta} + \widetilde{x}_2(2). \end{cases}$$

The effort level $\varkappa_1$ is a local maximum in the interval $[\max\{\frac{1}{\beta}, \widetilde{x}_2(2)-\frac{1}{\beta}\}, \frac{1}{\beta}+\widetilde{x}_2(2)]$ if and only if

$$\pi v_1 \left.\frac{\partial p_1(x_1, \widetilde{x}_2(2))}{\partial x_1}\right|_{x_1=\varkappa_1} = \varkappa_1 \Leftrightarrow \pi = \frac{2}{\beta v_1}\varkappa_1.$$

Observe that $\pi \leq 1$ if and only if

$$\varkappa_1 \leq \frac{\beta v_1}{2} = \widetilde{x}_1(2),$$

which holds automatically because of (17) and (19).

We now need to make sure that $\varkappa_1$ is also a global maximum, which entails discarding other candidates:

(i) Consider first the case $x_1 < \min\{\frac{1}{\beta}, \widetilde{x}_2(2) - \frac{1}{\beta}\}$. Denote as x_1' the candidate for global optimum in that interval when $\pi = \frac{2}{\beta v_1}\varkappa_1$. This effort must satisfy

$$(1-\pi)\beta\frac{v_1}{2} = c'(x_1') \Rightarrow x_1' = \beta\frac{v_1}{2} - \varkappa_1 = \beta\frac{v_1}{2} - \frac{1}{\beta} - \frac{\widetilde{x}_2(2)}{2}.$$

For x_1' to be a global maximum, it must satisfy $x_1' \in (0, \min\{\frac{1}{\beta}, \widetilde{x}_2(2) - \frac{1}{\beta}\})$. For $x_1' > 0$, it must be that

$$v_1 > \frac{2}{\beta}\varkappa_1 = \frac{2}{\beta^2} + \frac{\widetilde{x}_2(2)}{\beta}.$$

On the other hand, $x_1' < \min\{\frac{1}{\beta}, \widetilde{x}_1(2) - \frac{1}{\beta}\}$ requires

$$v_1 < \min\{\frac{4}{\beta^2} + \frac{\widetilde{x}_2(2)}{\beta}, 3\frac{\widetilde{x}_2(2)}{\beta}\}.$$

These bounds together imply that v_1 should be such that

$$\frac{2}{\beta^2} + 2\frac{\widetilde{x}_2(2)}{\beta} < v_1 < \min\{\frac{4}{\beta^2} + \frac{\widetilde{x}_2(2)}{\beta}, 3\frac{\widetilde{x}_2(2)}{\beta}\}.$$

However, straightforward calculations show that for these bounds to be compatible,

$$\frac{2}{\beta} < \widetilde{x}_2(2) < \frac{2}{\beta}, \tag{20}$$

leading to a contradiction. Hence, x_1' is not a maximum in the interval $[0, \widetilde{x}_2(2) - \frac{1}{\beta})$.

(*ii*) For $x_1 \in [\widetilde{x}_2(2) - \frac{1}{\beta}, \frac{1}{\beta})$ the local maximum would be $\widetilde{x}_1(2)$ as the marginal benefit of effort for player 1 would be again $\frac{\beta}{2}v_1$. Condition (17) however ensures that $\widetilde{x}_1(2) > \frac{1}{\beta}$, leading to a contradiction.

(*iii*) Because the payoff function is decreasing in the interval $x_1 \in [\frac{1}{\beta}, \widetilde{x}_2(2) - \frac{1}{\beta})$, the candidate local maximum here is $x_1' = 1/\beta$. Note that the payoff function has a kink at that level of effort. So for x_1' to be a local maximum, it must be that $u_1'(\frac{1}{\beta}, \varkappa_2) \geq 0$ when approaching $1/\beta$ from below. This in turn requires

$$(1-\pi)\frac{\beta}{2}v_1 \geq c'(\frac{1}{\beta}) \Leftrightarrow \pi \leq 1 - \frac{2}{\beta^2 v_1} \Leftrightarrow \varkappa_1 \leq \widetilde{x}_2(2) - \frac{1}{\beta},$$

leading to a contradiction since we impose $c(\widetilde{x}_2(2)) < v_2$ which is equivalent to $\varkappa_1 > \widetilde{x}_2(2) - \frac{1}{\beta}$.

(*iv*) For $x_1 \geq \frac{1}{\beta} + \widetilde{x}_2(2)$ the payoff function is again decreasing. The candidate local maximum is thus $x_1' = \frac{1}{\beta} + \widetilde{x}_2(2)$. The payoff function is again non-differentiable here so for this to be the optimum $u_1'(\frac{1}{\beta} + \widetilde{x}_2(2), \varkappa_2) \geq 0$ when approached from below $\frac{1}{\beta} + \widetilde{x}_2(2)$. This in turn requires

$$\pi\frac{\beta}{2}v_1 \geq c'(\frac{1}{\beta} + \widetilde{x}_2(2)) \Leftrightarrow \varkappa_1 \geq \frac{1}{\beta} + \widetilde{x}_2(2),$$

which contradicts the very definition of $\varkappa_1$. Since we have exhausted all other candidates for a global maximum, $\varkappa_1$ is the best response to $\varkappa_2$

The final step of the proof is to show that no player $i \geq 3$ would like to deviate and become active when players 1 and 2 are using strategies $\{\varkappa_1, \varkappa_2\}$. To do that we follow the same strategy as in the proof of Theorem 3.2 in Alcalde and Dahm (2010) as our success function also satisfies the monotonicity and anonymity properties they imposed on their CSF.[16]

More specifically, consider the case where some player $i \geq 3$ plays some pure strategy $x' > 0$. Note that our CSF implies that the expected winning probability of player i in that case satisfies

$$\pi p_i(\varkappa_1, \widetilde{x}_2(2), x') + (1-\pi)p_i(\varkappa_1, 0, x') = \pi p_2(\varkappa_1, x', \widetilde{x}_2(2)) + (1-\pi)p_2(\varkappa_1, x', 0) \leq p_2(\varkappa_1, x'),$$

where, to simplify notation, we are leaving out $\{0\}_{j \neq i, j \geq 3}$, the vector of zero efforts of all contestants different from players 1, 2 and i. The equality comes from applying

[16] Monotonicity is one of the axioms characterizing our CSF (Cubel and Sanchez-Pages, 2016). Anonimity is satisfied because (2) assumes in addition that impact functions are identical across players.

the anonymity property of the CSF and the inequality from applying monotonicity. Given that $v_2 \geq v_i$ for $i \geq 3$ then

$$\begin{aligned} u_i(\varkappa_1, \varkappa_2, x') &= v_i[\pi p_i(\varkappa_1, \widetilde{x}_2(2), x') + (1-\pi) p_i(\varkappa_1, 0, x')] - c(x') \\ &\leq v_2 p_2(\varkappa_1, x') - c(x') \\ &= u_2(\varkappa_1, x') \leq u_2(\varkappa_1, \varkappa_2) = 0, \end{aligned}$$

where the very last inequality holds from the fact that $\varkappa_2$ is player 2's best response to $\varkappa_1$. Hence, player $i \geq 3$ cannot gain from becoming active. ■

6. Conclusion

In this paper we have offered the first systematic study of multi-player contests where winning probabilities depend on the difference between participants' effective efforts. These contests are well suited to describe a wide variety of situations where a set of agents compete for an object or award. These include military combat, union-firm conflicts, influence activities within organization and political lobbying within federal countries. We have shown that there exists a natural connection between these contests and rank-order tournaments à la Lazear and Rosen (1981).

Our main result is that the non-existence of pure strategy equilibria and the full preemption phenomenon observed in the previous literature rested critically on the assumption of full linearity. This was to be expected. The lack of robustness of predictions under linear costs has been shown to apply to other families of contests. That said, partial preemption may take place in equilibrium when full linearity is abandoned. This is because the difference-form CSF can award a positive winning probability to inactive players. But we see this result as realistic. In many real-world environments, some contenders remain inactive: Not all countries engage in warfare nor all employees engage in influence activities. Partial preemption is indeed a real-world phenomenon.

Further research should deepen our understanding of difference-form contests. One avenue is the study of group contests, which we undertake in a companion paper (Cubel and Sanchez-Pages, 2014). Another open avenue would be along the lines of Baye and Hoppe (2003), namely, to study the strategic equivalence of these contests with other well-known competitive games. Finally, a full characterization of mixed equilibria remains pending. We would have liked to study the limiting behavior of these equilibria when the sensitivity of effort differentials becomes arbitrarily large. Che and Gale (2000) did this under full linearity and showed that equilibrium payoffs converge to those of the all-pay auction (Hillman and Riley, 1989; Baye, Kovenock and de Vries, 1996).[17] But the problem becomes quite intractable for the generalized

[17] Alcalde and Dahm (2010) show this for a class of contests that contains Tullock contests as a special case; see also Ewerhart (2017) for Tullock contests.

case. Still, the system of equations characterizing the staggered mixed equilibrium in Section 5.2 suggests that the points where players put positive mass must become increasingly close as the sensitivity increases, even closer for higher levels of effort. This is in consistent with the convex cumulative distribution functions characterizing the mixed strategy equilibrium of the all-pay auction under convex costs (Kaplan and Wettstein, 2006).

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